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On an integrable stochastic Volterra lattice

S De Lillo† and V V Konotop‡§

† Dipartimento di Fisica, Universita di Perugia, Perugia 2-00185, Italy

‡ Department of Physics, University of Madeira, Praça do Município, P-9000 Funchal, Portugal

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Abstract. We introduce an integrable inhomogeneous generalization of the Volterra lattice, allowing one to treat linear and nonlinear multiplicative noises and to discuss statistical properties of the exact solutions.

1. Introduction

Although during the last few years a great deal of attention has been paid to nonlinear systems perturbed by random forces (see e.g. [1–3] and references therein) the behaviour of the most recent is still not well understood. Probably the most important problem which still remains to be resolved is the choice of an adequate set of parameters describing a nonlinear random system. For instance, it is well known that various momenta of a nonlinear field display dynamics which, in generic situations, drastically differs from the dynamics of the field itself. The first examples of this phenomenon were reported in [4, 5] (see also [1]). There it has been shown that integrable addends to the nonlinear Schrödinger equation [4] and to the Korteweg–de Vries (KdV) equation [5] result in random motion of a soliton without a change in its shape, while respective mean fields are Gaussian wavepackets spreading with time. In principle, by using the transition to a moving frame it is possible to indicate a general class of nonlinear evolution stochastic equations which manifests Gaussian spreading of the mean field but distorsionless propagation of the field itself [1]. If a system is integrable by means of the inverse-scattering technique its generalization allowing the inclusion of inhomogeneous terms can be found by requiring a spectral parameter to depend on time. This also allows observation of the phenomena mentioned above, in the case of an integrable inhomogeneous discrete version of the nonlinear Schrödinger equation [6, 7], the homogeneous version of which is known as the Ablowitz–Ladik model [8]. As has been shown in [7] a very useful technical tool in the analysis of inhomogeneous integrable lattices is a gauge transformation. We also use this approach below.

As regards discrete systems we must emphasize one more aspect of the problem. Namely, even in a regular case discretization is not an umbiguious procedure. In the stochastic case even less is known about the relation between the continuum evolution equation and its discrete version. In particular, to the best of the authors' knowledge the discrete integrable version of the stochastic KdV equation introduced by Wadati [5] has not been investigated and reported so far.

§ Also at: Center of Science and Technology of Madeira (CITMA), Rua da Alfândega, 75-5°, Funchal P-9000, Portugal.

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The purpose of this paper is to introduce and study an integrable stochastic Volterra (ISV) lattice, which in the continuum limit reduces to the stochastic KdV equation [5]. We choose the integrable model because its great advantage is that the field can be explicitly expressed through the random field. This allows us to compare field configuration with its averaged values and adequacy analyse various stochastic characteristics.

The paper is organized as follows. The integrable inhomogeneous lattice is introduced in section 2. There we also discuss the continuum limit of the model and represent its soliton solution. Section 3 is devoted to statistical characteristics of a one-soliton solution. The outcomes are summarized in the conclusion.

2. The integrable inhomogeneous Volterra model and its continuum limit

Let us start with a UV-pair as follows

$$U_n = \begin{pmatrix} \lambda(t) & v_n(t) \\ -1 & 0 \end{pmatrix} \tag{1}$$

$$Vn = \begin{pmatrix} v_n(t) + \gamma(t)n & v_n(t)\left(\lambda(t) - \frac{p_{n+1}(t)}{\lambda(t)}\right) \\ -\lambda(t) + \frac{p_n(t)}{\lambda(t)} & v_{n-1}(t) - \lambda^2(t) + \gamma(t)(n-1) + p_n(t) \end{pmatrix}. \tag{2}$$

Here $\lambda(t)$ is a spectral parameter which is allowed to be dependent on time and $\gamma(t)$ is an arbitrary (in particular, random) function of time. Then the zero-curvative condition (see e.g. [10])

$$\dot{U}_n + U_n V_n - V_{n+1} U_n = 0 \tag{3}$$

results in a system of equations

$$\dot{\lambda} - \gamma\lambda - \frac{q}{\lambda} = 0 \tag{4}$$

$$\dot{v}_n + v_n(v_{n-1} - v_{n+1}) - 2\gamma v_n + (p_n - p_{n+1})v_n = 0 \tag{5}$$

$$p_n v_n - p_{n+2} v_{n+1} = -q \tag{6}$$

where $q \equiv q(t)$ can be an arbitrary function of time (but not of the site number n). As is evident, at $\gamma \equiv 0$ and $q \equiv 0$ the spectral parameter λ is a constant and (5) is reduced to the well known Volterra equation [9].

In order to understand the physical meaning of the introduced functions $\gamma(t)$, $p(t)$ and $q(t)$ let us consider the continuum limit of (5), (6) using the scaling as follows: $x = \epsilon n$, $\gamma(t) = \epsilon^5 \tilde{\gamma}(t)$, $q(t) = \epsilon^5 \tilde{q}(t)$,

$$v_n = 1 - \epsilon^2 \tilde{v}(x) \qquad p_n = p(t)[1 + \epsilon^2 \rho_1(x) + \epsilon^3 \rho_2(x) + \epsilon^4 \rho_3(x) + \cdots] \tag{7}$$

where ϵ is a small parameter (the functions $v(x)$ and $\rho(x)$ also depend on time). By straightforward algebra we obtain from (6) the relations

$$\frac{\partial \rho_1}{\partial x} = \frac{1}{2}\frac{\partial \tilde{v}}{\partial x} \tag{8}$$

$$\frac{\partial \rho_2}{\partial x} = -\frac{1}{4}\frac{\partial^2 \tilde{v}}{\partial x^2} \tag{9}$$

$$\frac{\partial \rho_3}{\partial x} = \frac{3}{8}\frac{\partial \tilde{v}^2}{\partial x} + \frac{\tilde{q}(t)}{p(t)}. \tag{10}$$

Then (17) yields

$$\frac{\partial \tilde{v}}{\partial t} - \epsilon\left(2 - \frac{p(t)}{2}\right)\left(\frac{\partial \tilde{v}}{\partial x} - \epsilon^2 \tilde{v}\frac{\partial \tilde{v}}{\partial x} - \frac{1}{6}\epsilon^3 \frac{\partial^3 \tilde{v}}{\partial x^3}\right) + 2\epsilon^3 \tilde{\gamma} + \epsilon^3 \frac{\tilde{q}(t)}{2p(t)} = 0. \tag{11}$$

By introducing new variables

$$X = x + 2\epsilon\tau \qquad T = \frac{\epsilon^3}{3}\tau \tag{12}$$

$$\tau = t - \tfrac{1}{4}\int_0^t p(t')\,\mathrm{d}t' \tag{13}$$

equation (11) is reduced to

$$\frac{\partial\tilde{v}}{\partial T} - \frac{\partial^3\tilde{v}}{\partial X^3} + 6\tilde{v}\frac{\partial\tilde{v}}{\partial X} + \gamma'(T) + q'(T) = 0 \tag{14}$$

here $\gamma'(T) = 6\tilde{\gamma}(t)/\tau$ and $q'(T) = \frac{3}{2\tau}\frac{\tilde{q}(t)}{p(t)}$. Thus the last term in (5) and the inhomogeneity in (6) in the continuum limit give additive perturbations. Considering $\gamma'(T)$ and $q'(T)$ as random functions one arrives at the KdV equation with additive noise studied by Wadati [5]. In this context it is interesting to note that $\gamma(t)$ and $p(t)$ in (5) and (6) can be interpreted as the linear and nonlinear multiplicative noise, respectively.

In this paper we concentrate on a particular case of the system (4)–(6), where $q \equiv 0$, subject to the boundary conditions as follows,

$$\lim_{|n|\to\infty} v_n = v(t) = \mathrm{e}^{2\Gamma(t)} \qquad \lim_{|n|\to\infty} p_n = p(t) \tag{15}$$

where

$$\Gamma \equiv \Gamma(t) = \int_0^t \gamma(t')\,\mathrm{d}t' \tag{16}$$

and $p(t)$ is an arbitrary function of time. Note that the spectral parameter in the case at hand is given by $\lambda = \lambda_0\mathrm{e}^{\Gamma}$ (λ_0 being the initial value of the spectral parameter). The relation (6) can be resolved with respect to p_n

$$p_n = p(t)\prod_{k=1}^{\infty}\frac{v_{n-2k}}{v_{n+1-2k}}. \tag{17}$$

Thus the term with p_n describes nonlocal interactions in the lattice.

In order to find soliton solutions of (5) and (17) we first note that (5) is gauge equivalent to the equation

$$\dot{u}_n + vu_n(u_{n-1} - u_{n+1}) + u_n(p_n - p_{n+1}) = 0 \tag{18}$$

where u_n is linked with v_n: $u_n = v_n/v(t)$, and hence is subject to the boundary conditions $\lim_{|n|\to\infty} u_n = 1$. The gauge equivalence is provided by the matrix $G(n) = v^{n/2}\mathrm{diag}(1, v^{-1/2})$. The respective $\tilde{U}_n$ matrix, $\tilde{U}_n(\tilde{\lambda}) = G^{-1}(n+1)U_n(\lambda)G(n)$, where $\tilde{\lambda} = \lambda/\sqrt{v}$ (= constant at $q = 0$), coincides with the U-matrix for the homogeneous Volterra lattice (for the last one see e.g. [10]). This allows us to use the well known results on the Volterra lattice and after standard computing the temporal dependence of the scattering data to write down the one-soliton solution of equation (5) in the form

$$v_n = v(t)\frac{\cosh\Theta_{n+1}\cosh\Theta_{n-2}}{\cosh\Theta_n\cosh\Theta_{n-1}} \tag{19}$$

where

$$\Theta_n = -K[n - n_0 + X(t)] \tag{20}$$

$$X(t) = \frac{\tanh(K)}{2K}P(t) - \frac{\sinh(2K)}{K}V(t) \tag{21}$$

$$P(t) = \int_0^t p(t')\,\mathrm{d}t' \tag{22}$$

$$V(t) = \int_0^t v(t')\,\mathrm{d}t' \tag{23}$$

K is a positive constant defining soliton width and amplitude, and n_0 is a constant which plays a part of the initial position of the soliton (in what follows it will be taken equal to zero). As is evident $X(t)$ plays a role of the coordinate of the soliton centre.

3. Statistical characteristics of the one-soliton solution

Before going into details of the statistics of the one-soliton dynamics we make some general comments about the solution (19).

The soliton (19) can be viewed as an excitation against a background $v(t)$ which is time dependent. It then follows from (19)–(23) that the linear multiplicative inhomogeneity affects both the soliton velocity and the background. In particular, if γ is a negative constant (i.e. it describes dissipation), and respectively $v = \exp(-2|\gamma|t)$, the soliton velocity also decreases in accordance with the exponential law (note that this result is exact and hence is valid at any t).

The nonlinear multiplicative inhomogeneity affects only the velocity of the soliton. In particular, at constant p the velocity shift is given by $\frac{\tanh K}{2K}p$.

Periodic $p(t)$ results in oscillation of the soliton while periodic $\gamma(t)$ leads to both oscillations and a change of the mean value of the velocity.

In this paper we are mainly interested in a situation when $\gamma(t)$ and $p(t)$ are random functions. More precisely we consider them to Gaussian coloured noises with the characteristics as follows,

$$\langle\gamma(t)\rangle = \langle p(t)\rangle = 0 \tag{24}$$

$$\langle\gamma(t)\gamma(t')\rangle = \frac{D_\gamma}{2\tau_\gamma}\exp\left(-\frac{|t-t'|}{\tau_\gamma}\right) \tag{25}$$

$$\langle p(t)p(t')\rangle = \frac{D_p}{2\tau_p}\exp\left(-\frac{|t-t'|}{\tau_p}\right). \tag{26}$$

Here positive constants $D_{\gamma,p}$ characterize intensities of fluctuations and $\tau_{\gamma,p}$ are respective autocorrelation radii (at $\tau_{\gamma,p} \to 0$ the introduced distributions reduce to white noises). The random processes $\gamma(t)$ and $p(t)$ are considered to be mutually uncorrelated: $\langle\gamma(t)p(t)\rangle = 0$.

It follows from the general form of the solution (19) that the soliton evolution can be split into three different processes: homogeneous background oscillations, diffusion caused by the nonlinear multiplicative noise $p(t)$ and diffusion due to the linear multiplicative noise $v(t)$. The first of the processes is originated by the linear multiplicative noise and is associated with exponential growth of the mean value of the background:

$$\langle v(t)\rangle = \exp\{2D_\gamma[t + \tau_\gamma(\mathrm{e}^{-t/\tau_\gamma} - 1)]\}. \tag{27}$$

These amplitude fluctuations are modulated by the function u_n which describes soliton fluctuations. Meanwhile the average soliton solution and its higher momenta grow, together with the momenta of the background and hence are not adequate statistical characteristics. This is why, in what follows, we concentrate on the stochastic dynamics of a soliton centre which seems to be the most interesting physical value.

As mentioned above the diffusion of the soliton is described by two uncorrelated components. One of them, due to the process $p(t)$ in the limit $\tau_p \to 0$ is nothing but the Brownian motion. The random motion originated by the linear multiplicative noise $v(t)$ is characterized by the mean velocity $\mathrm{d}X(t)/\mathrm{d}t = -(\sinh(2K)/K)\langle v(t)\rangle$. Thus as a result of fluctuations the acceleration of the soliton occurs [see (27)] which reflects the fact that the mean amplitude of the background grows. The growth of the mean value of the background can be considered as a peculiarity of the discrete problem since the mean value of the background in the respective continuum model (14) is zero. Moreover the two-point correlator of the background in the discrete case displays exponential growth while in the continuum approximation the growth of the background dispersion if governed by the power law. This is a reflection of a general fact that the discreteness introduces a new spatial scale in the problem and as a consequence changes the statistics of the response of the system on the effect of stochastic forces.

The mean-square displacement is given by

$$\langle X^2\rangle = \frac{\sinh^2(2K)}{12D_\gamma^2 K^2}\mathrm{e}^{8D_\gamma t - 6D_\gamma \tau_\gamma} + \frac{\tanh^2 K}{4K^2} Dp^t. \tag{28}$$

Hereafter, for the sake of simplicity, we keep only the leading term in the expansion with respect to small parameters $\exp(-t/\tau_{\gamma,p})$.

Fluctuations of the background and the soliton diffusion being originated by the same stochastic process are correlated. For t large enough one obtains

$$\langle v(t)X(t)\rangle \approx \frac{\sinh(2K)}{6KD_\gamma}\exp[2D_\gamma(4t + 3\tau_\gamma(\mathrm{e}^{-t/\tau_\gamma} - 1))]. \tag{29}$$

We finally consider the correlation between the nonlinear inhomogeneity $p(t)$ and the soliton centre coordinate $X(t)$. From (21) we obtain the correlator at different times

$$\langle p(t_1)X(t_2)\rangle = -\frac{\tanh(K)}{2K}\langle p(t_1)P(t_2)\rangle. \tag{30}$$

We first consider the case $t_1 > t_2$. From (22) and (26) we obtain

$$\langle p(t_1)P(t_2)\rangle = \frac{D_p}{2}\left[\exp\left(\frac{t_2 - t_1}{\tau_p}\right) - \exp\left(-\frac{t_1}{\tau_p}\right)\right]. \tag{31}$$

We observe that the above correlator is exponentially vanishing as t_1 grows (t_2 being fixed), moreover it is also vanishing in the white-noise limit $\tau_p \to 0$ (t_1 and t_2 being fixed). This result tells us that for a truly uncorrelated process (white noise) the process at a given time cannot affect the soliton position at a previous time.

In the case $t_1 < t_2$ we obtain instead

$$\langle p(t_1)P(t_2)\rangle = \frac{D_p}{2}\left[2 - \exp\left(\frac{t_1 - t_2}{\tau_p}\right) - \exp\left(-\frac{t_1}{\tau_p}\right)\right] \tag{32}$$

which shows that the correlator is different from zero both in the case when t_2 grows (t_1 being fixed) and in the white-noise case, $\tau_p \to 0$. At $t_1 = t_2 = t$ (31) and (32) give the equal time correlator

$$\langle p(t)X(t)\rangle = -\frac{\tanh K}{4K}D_p(1 - \mathrm{e}^{-t/\tau_p}). \tag{33}$$

4. Conclusion

To conclude we have introduced an integrable stochastic Volterra lattice which in the continuum limit is reduced to the integrable stochastic KdV equation. Although in the continuum limit the two different processes $\gamma(t)$ and $q(t)$ give additive noises of formally the same kind, the respective terms in the discrete system are essentially different. Moreover, linear and nonlinear multiplicative noises in the lattice are reduced to additive noises in the respective continuum model. Only one of them, $\gamma(t)$, has been considered in detail in this paper. The spectral parameter of the associated linear problem is a non-Gaussian random process. The one-soliton solution of the problem is a localized excitation displaying random motion against a fluctuating background.

Although in this paper we have analysed the stochastic dynamics of the single soliton, some remarks about multisoliton solution are in order. As is well known, in the absence of the stochastic forces the structure of multisoliton solutions, at least in the assymptotic region, is rather completely described by the discrete spectrum of the associated linear problem and the trace of the soliton–soliton interactions exists only in phases. When stochastic forces are included in the consideration the situation is changed since solitons involved in the dynamics undergo a random number of mutual collisions. As a consequence of this the displacement of a single soliton is determined not only but is own random walk (i.e. its motion in the absence of other solitons but also by random phase shifts due to random collisions. Moreover, now the limit $t \to \infty$ does not correspond necessarily to spatial separation of solitons. Thus, although the formal expression of the multisoliton solution of the problem (4)–(6) can be directly obtained from the multisoliton solution of the conventional homogeneous Volterra lattice, the analysis of the 'fine structure' of the multisoliton solution is reduced to a study of a rather combersome stochastic problem (we leave that problem for further investigations).

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Appendix. LA-pair

An alternative way of solving integrable equations is to represent them in the form of a Lax pair. In the case at hand, however, the zero-curvative-condition representation seems to be drastically simpler. In order to illustrate this here we represent the LA-pair for the equation (18)

$$L_{nm} = \sqrt{u_n}\delta_{n,m+1} + \sqrt{u_m}\delta_{n+1,m} \tag{34}$$

$$A_{nm} = \tfrac{1}{2}v(t)\sqrt{u_n u_{n-1}}\delta_{n,m+2} - \tfrac{1}{2}v(t)\sqrt{u_m u_{m-1}}\delta_{n+2,m} - \tfrac{1}{2}a_{n,m} \tag{35}$$

where

$$a_{nm} = 0$$

at $n = m$, or $n - m = 2k + 1$ (k being an integer),

$$a_{nm} = (-1)^k \chi_{m+1} \prod_{q=0}^{k} \sqrt{\frac{u_{n+2q+2}}{u_{n+2q+1}}}$$

at $n - m = -2k$ (k being a positive integer),

$$a_{nm} = (-1)^{k-1} \chi_{m+1} \prod_{q=1}^{k-1} \sqrt{\frac{u_{n-2q+1}}{u_{n-2q+2}}}$$

at $n - m = 2k$ (k being a positive integer), and

$$\chi_m = -\frac{p(t)}{u_m u_{m+1}} \prod_{k=0}^{\infty} \frac{u_{m-2k}}{u_{m-1-2k}}.$$

Thus, while the L-matrix has the standard form [11], the A-matrix is much more complicated than in the absence of the nonlinear multiplicative inhomogeneity ($a_{nm} \equiv 0$).

It is interesting to note that by comparing the LA representation with (1)–(3) one can compute the inverse matrix $(L^{-1})_{nm}$.

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